INTERACTION OF RADIATION AND A RELATIVISTIC ELECTRON IN MOTIOM IN A CONSTANT MAGNETIC FIELD

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Abstract

This work examines the effect of multiple photon emission on the quantum mechanical state of an electron emitting synchrototron radiation and on the intensity of that radiation. Calculations are done with a variant of perturbation theory based on the use of extended coherent states. A general formula is derived for the number of emitted photons, which allows for taking into account their mutual interaction. A model problem is used to demonstrate the absence of the infrared catastrophe in the modified perturbation theory. Finally, the elecron density matrix is calculated, and the analysis of this matrix makes it possible to conclude that the degree of the electron's spatial localization increases with the passage of time if the electron is being accelerated.

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1. INTRODUCTION

The effect of radiation on the path of charged particles in a synchrotron has already been analysed (see, e.g., Ref.1 and the literature cited therein). The analysis is based on the classical Lorenz-Dirac equation or on the solution of a kinetic equation whose coefficients are the probabilities of quantum transitions between various stationary states of an electron moving in the magnetic field of the synchrotron. Calculations have shown that in the absence of focusing in the magnetic field there is increase in the radial fluctuations of the electron path and an increase in longitudinal fluctuations of the electron's momentum with the passage of time. In a focusing magnetic field, in the initial stages of electron acceleration, the presence of radiation leads to radiative damping, which damps radial and vertical oscillations the so-called radiative damping effect). Lately, research are focused on the analysis of equation of Lorenz- Dirac type in problem not necessarily related to synchrotron radiation (see, e.g., Refs. 2—4). New solutions of Lorenz- Dirac equations have been found for some special cases, and additional arguments from quantum electrodynamics are invoked to eliminate nonphysical solutions.

Despite the indisputable value of the results obtained by solving Lorenz-Dirac equations, it must be noted that some important properties of the states of a particle emitting radiation, properties that are not directly related to the path and do not directly influence the above effects — like an increase in radial fluctuations of the path — are excluded from these results. This is true, in particular, of the evolution of the particle's wave packet, which affects the radiation and hence the radiative friction and the path.

The present work is an attempt to use a modified perturbation- theory approach to examine the effect of multiple photon emission on the evolution of the wave packet of a particle, in particular when the particle emits synchrotron radiation. Most papers devoted to the quantum mechanical theory of the synchrotron radiation ignore this aspect. It is usually assumed that the particle emitting radiation has a wave function given by the solution of the Dirac equation. However, if the emitted radiation is taken into account, the particle is only a part of of the quantum mechanical system and its state cannot be described with the completeness that is possible in principle in quantum theory.

A common approach to describing the states of particles that are members of a large system is to use the concept of the density matrix. This work demonstrates that the evolution of the density matrix suggests that a particle goes with the passage of time into states that are more and more localized, with the particle motion described by the laws of classical mechanics with ever-increasing accuracy. Thus we have additional support for the validity of using the Lorenz-Dirac equation along with a clearer undestanding of the incomleteness of the physical picture described by this equation.

The problem of the structure of the wave packets of emitting particles is related to the classical model of a distributed electron studied by Lorenz (see, e.g., Ref. 5). In quantum electrodynamics this model leads to the well-known problem of ultraviolet divergence, encountered in the calculations of the mass, charge, and energy of an elementary particle. Furthermore, the renormalization of charge in quantum electrodynamics reveals the internal inconsistency of the traditional Feynman formulation of perturbation theory (see, e.g., Ref. 6). It would be useful to follow the changes in the difficulties encountered by classical electrodynamics initiated by changes in the perturbation theory, to establish which of the above problems is invariant, so to say. It might turn out that in the modified theory some of these problems can be resolved without resorting to additional hypotheses. This might then lead to a new direction in the development of quantum electrodynamics and the theory of quantized fields in general. The presen work uses a model to show that at least in relation to the infrared catastrophe, the adopted modification in the theory does not lead to problems characteristic of the traditional form of the theory. Calculations are based on a general formula that describes the mutual interaction of the emitted photons as a manifestation of the nonlinearity inherent in quantum electrodynamics.

2. EMISSION OF PHOTONS BY A CLASSICAL CHARGED PARTICLE

We write the Hamiltonian describing the interaction of a free electromagnetic field and a particle carrying an electric charge Z (here we use atomic units: $\hbar = 1$ and $|e| = 1$):

$$\widehat{H}_{int} = -\frac{1}{c}\int \mathbf{j}\widehat{\mathbf{A}}dV, \tag{1}$$

where the current density vector $\mathbf{j}$ is a function of coordinates and time. The vector potentual operator is specified in a three-dimensional transverse gauge,

$$\widehat{\mathbf{A}} = \sum_{\alpha,\mathbf{q}} g_q\{\widehat{f}_{\alpha\mathbf{q}}\mathbf{e}_{\alpha\mathbf{q}}e^{i\mathbf{qr}} + \widehat{f}^{\dagger}_{\alpha\mathbf{q}}\mathbf{e}^{*}_{\alpha\mathbf{q}}e^{-i\mathbf{qr}}\} \tag{2}$$

as a standard linear form in the creation and annihilation operators ($\hat{f}^\dagger_{\alpha\mathbf{q}}$ and $\hat{f}_{\alpha\mathbf{q}}$ for photons in states with polarization $\alpha(\alpha = 1, 2)$, momentum $\mathbf{q}$, and energy $\omega = cq$. The polarization vectors $\mathbf{e}_{\alpha\mathbf{q}}$ have unit length and are orthogonal to $\mathbf{q}$. The coupling constants $g_q = (2\pi c^2/\omega\Omega)^{1/2}$ contain the normalization volume Ω, which does not enter into the final expressions and thus can be put equal to unity.

The total Hamiltonian is the sum of the free photon Hamiltonian

$$\hat{H}_0 = \sum_{\alpha\mathbf{q}} \omega\, \hat{f}^\dagger_{\alpha\mathbf{q}} \hat{f}_{\alpha\mathbf{q}} + const.$$

and the Hamiltonian (1). We pass to the interaction picture for field operators, for example,

$$\hat{A}(t) = e^{i\hat{H}_0 t} \hat{A} e^{-i\hat{H}_0 t}.$$

The equation describing the evolution of the wave vector of the photon field, $|t)$ in the interaction picture,

$$i\frac{d}{dt}|t) = \hat{H}_{int}(t)|t), \tag{3}$$

has in the given case an exact solution in the form of the direct product of photon coherent states,

$$|t) = \prod_{\alpha,\mathbf{q}} \exp[-i\chi_{\alpha,\mathbf{q}} - \hat{f}_{\alpha\mathbf{q}} Q^*_{\alpha\mathbf{q}} + \hat{f}^\dagger_{\alpha\mathbf{q}} Q_{\alpha\mathbf{q}}]|t_0), \tag{4}$$

where the initial state vector coinsides, to within the arbitrary phase factor, with the vacuum state of the photon field:

$$|t_0) = e^{i\phi_0}|vac), \phi_0 = const.,$$

$$Q_{\alpha\mathbf{q}}(t) = i\frac{g_q}{c}\int_{t_0}^{t} dt' \mathbf{e}^*_{\alpha\mathbf{q}} \mathbf{j}_\mathbf{q}(t') e^{i\omega t'},$$

$$\chi_{\alpha\mathbf{q}}(t) = \int_{t_0}^{t} \mathrm{Im}[Q_{\alpha\mathbf{q}}(t')\dot{Q}^*_{\alpha\mathbf{q}}(t')]dt',$$

and $\mathbf{j}_\mathbf{q}(\mathbf{t})$ is the Fourier transform of the current density.

Using the exact solution (4), we can calculate all quantities of interest. For instance, the mean number of photons created by time t is given by

$$n^{(0)}_{\alpha\mathbf{q}}(t) = |Q_{\alpha\mathbf{q}}(t)|^2. \tag{5}$$

Note that this formula yields the mean number of emitted photons only as $t \to \infty$, since creation of a photon requires a time interval c/q long, which tends to infinity as $q \to 0$. In what folows we use the interpretation of $n_{\alpha\mathbf{q}}$.

Suppose that the photons are emitted by a point particle carrying electric charge Z and moving along a path $\mathbf{r} = \mathbf{r}_0(t)$. Then

$$\mathbf{j_q}(t) = Z\mathbf{v}_0(t)\exp[-i\mathbf{q}\mathbf{r}_0(t)],$$

where $\mathbf{v}_0 = \dot{\mathbf{r}}_0(t)$. After summing over over polarizations we can reduce the time derivative of the number of photons (as $t_o \to -\infty$) to the form

$$\frac{d}{dt}\sum_{\alpha=1,2} n^{(0)}_{\alpha\mathbf{q}}(t) = \frac{Z^2}{c^2}g_q^2 \int\limits_{-\infty}^{\infty} d\tau[\mathbf{v}_0(t-|\tau|/2+\tau/2)\mathbf{v}_0(t-|\tau|/2-\tau/2)-$$

$$\frac{1}{q^2}(\mathbf{q}\mathbf{v}_0(t-|\tau|/2+\tau/2))(\mathbf{q}\mathbf{v}_0(t-|\tau|/2-\tau/2))]*$$

$$\exp[i\omega\tau - i\mathbf{q}(\mathbf{r}_0(t-|\tau|/2+\tau/2) - \mathbf{r}_0(t-|\tau|/2-\tau/2))]. \tag{6}$$

Applying this equation to the case of synchrotron radiation, we obtain

$$\frac{d}{dt}\sum_{\alpha} n^{(0)}_{\alpha\mathbf{q}} = Z^2\frac{v_0^2}{c^2}g_q^2 \int\limits_{-\infty}^{\infty} d\tau[\cos\omega_0\tau - \frac{1}{2}\cos^2\Theta(\cos\omega_0\tau + \cos((2t-|\tau|)\omega_0))]*$$

$$\exp\{i\omega\tau - 2iqR\cos\Theta\,\sin\frac{\omega_0\tau}{2}\,\cos(\omega_0(t-|\tau|/2))\}, \tag{7}$$

where $\omega_0 = e\mathrm{H}_0/\gamma mc, v_0 = R\omega_0, \mathrm{H}_0$ is the magnetic field strength, R is the orbit's radius, m is the particle mass, and γ is the Lorenz factor. The angle Θ is the inclination of the vector $\mathbf{q}$ to the orbital plane.

The expression (7) is periodic in time, with period $T_0 = 2\pi/\omega_0$. Averaging over one period, we obtain

$$\frac{d}{dt}\sum_{\alpha}\overline{n^{(0)}_{\alpha\mathbf{q}}} = Z^2\frac{v_0^2}{c^2}g_q^2 \int\limits_{-\infty}^{\infty} d\tau\, e^{i\omega\tau}\left[\frac{1}{2}\cos^2\Theta\; J_2\left(2qR\cos\Theta\,\sin\frac{\omega_0\tau}{2}\right) + \right.$$

$$\left.\cos\omega_0\tau\left(1-\frac{1}{2}\cos^2\Theta\right)\; J_0\left(2qR\cos\Theta\,\sin\frac{\omega_0\tau}{2}\right)\right]. \tag{8}$$

Next, we allow for the fact that for any periodic function $F(\tau)$,

$$\int\limits_{-\infty}^{\infty} e^{-i\omega\tau}F(\tau)d\tau = \sum_{n=-\infty}^{\infty} e^{in\omega T_0}\int\limits_{0}^{T_0} e^{i\omega\tau}F(\tau)d\tau, \tag{9}$$

where the sum of exponentials can be transformed into a sum of delta functions:

$$\sum_{n=-\infty}^{\infty} e^{in\omega T_0} = \frac{2\pi}{T_0} \sum_{n'=-\infty}^{\infty} \delta(\omega - n'\omega_0).$$

Combining this with (9), we can transform (8) to the following form:

$$\frac{d}{dt}\sum_{\alpha}\overline{n_{\alpha\mathbf{q}}^{(0)}} = Z^2\frac{v_0^2}{c^2}g_q^2\sum_{n=0}^{\infty}\frac{1}{\pi}\int_0^{\pi} dx\, e^{2inx}\{\cos 2x\, J_0(2qR\,\cos\Theta\,\sin x)(1-\frac{1}{2}\cos^2\Theta)+$$

$$\frac{1}{2}\cos^2\Theta\, J_2(2qR\,\cos\Theta\,\sin x)\}2\pi\delta(\omega - n\omega_0). \tag{10}$$

Since $\omega = cq > 0$, the sum in (10) goes from 0 to ∞.

Next we have[7]

$$\int_0^{\pi} e^{2i\mu x}\, J_{2\nu}(2a\,\sin x)dx = \pi e^{i\pi\mu} J_{\nu-\mu}(a)\, J_{\nu+\mu}(a),$$

and the recurrence formulas

$$J_{n+1}(z) + J_{n-1}(z) = \frac{2n}{z}J_n(z);$$

$$J_{n-1}(z) = \frac{n}{z}J_n(z) + J_n'(z);$$

$$J_{n+1}(z) = \frac{n}{z}J_n(z) - J_n'(z).$$

As a result, Eq. (10) becomes

$$\frac{d}{dt}\sum_{\alpha}\overline{n_{\alpha\mathbf{q}}^{(0)}} = Z^2 g_q^2\sum_{n=0}^{\infty}2\pi\delta(\omega - n\omega_0)[\mathrm{tg}^2\Theta\, J_n(\frac{nv}{c}\cos\Theta) + \frac{v_0^2}{c^2}J_n'^2(\frac{nv}{c}\cos\Theta)]. \tag{11}$$

Equation (11) can be used, in particular, to obtain the well- known Schott formula. Thus, for the mean intensity of synchrotron radiation the semiclassical theory yields results that coinside with classical results. The semiclassical theory provides additional information (in comparison to that provided by classical electrodynamics) only in the sence that it makes it possible to calculate the fluctiations in the number of the emitted photons, their mean energy, and total momentum. Similar results can be obtained for the case in which a charged particle moves along an arbitrary path[8]. In all cases we at classical formulas for the mean intensity of radiation emitted by the particle. Moreover, calculations of the mean electromagnetic field that accompanishes a charged particle moving *in vacuo* lead to well-known expressionsfor the retarded potentials[9]. This agreement between the semiclassical and classical theories forms the basis for a more accurate quantum mechanical theory of interaction of radiation and an emitting particle.

3. QUANTUM MECHANICAL THEORY

We consider the interaction of an electron and the radiation emitted by that electron. We pass to the furry representation and write the wave operator of the electron in the form of an expansion in the stationary states of type (A2) (see the Appendix):

$$\widehat{\psi} = \sum_{\xi} \widehat{d}_{\xi} \psi_{\xi},$$

where we have excluded the antiparticle operators, since allowing for the contribution of particle-antiparticle intermediate states leads only to small corrections to the phenomena considered. The creation and annihilation operators, $\widehat{d}_{\xi}^{\dagger}$ and $\widehat{d}_{\xi}$, must obey the standard Fermi commutation relations. The current density operator is approximately (without allowing for electron-positron pair contributions) given by

$$\widehat{j}_a(t) = c\widehat{\psi}^{\dagger}\alpha_a\widehat{\psi} = c\sum_{\xi\xi'} \widehat{d}_{\xi}^{\dagger}\widehat{d}_{\xi'}\psi_{\xi}^{*}(\mathbf{r})\alpha_a\psi_{\xi'}(\mathbf{r})e^{i(E_{\xi}-E_{\xi'})t} \tag{12}$$

(from now on $a, b| = 1, 2, 3$ label the projections of vectors on the Cartesian coordinate axes).

We construct the operator

$$\widehat{\mathbf{j}}_{\mathbf{q}}^{(0)}(t) = Z\dot{\mathbf{r}}_0(\mathbf{q}, t)e^{-i\mathbf{q}\mathbf{r}_0(\mathbf{q},t)}\widehat{\rho}_{\mathbf{q}}^{(0)}, \tag{13}$$

where $\mathbf{r}_0(\mathbf{q}, t)$ is a vector (which needs to be determined) that depends on the momentum transfer $\mathbf{q}$ and time t, and $\widehat{\rho}_{\mathbf{q}}^{(0)}$ is the "zeroth" density operator at time $t = 0$:

$$\widehat{\rho}_{\mathbf{q}}^{(0)} = \sum_{\mathbf{k},\sigma} \widehat{d}_{\mathbf{k}\sigma}^{\dagger}\widehat{d}_{\mathbf{k}+\mathbf{q},\sigma}.$$

Here $\widehat{d}_{\mathbf{k}\sigma}^{\dagger}$ and $\widehat{d}_{\mathbf{k}\sigma}$ are the creation and annihilation operators for an electron in a state with momentum $\mathbf{k}$ and a projection of the electron spin on the z axis that takes the values $\sigma = \pm\frac{1}{2}$. Note that the operator (13) is selected in a form that satisfies the charge conservation law.

We require that the running mean Fourier transform of the operator (12) coincide with the expectation value of (13):

$$(t|\widehat{\mathbf{j}}_{\mathbf{q}}(t)|t) = \dot{\mathbf{r}}_0(\mathbf{q}, t)e^{-i\mathbf{q}\mathbf{r}_0(\mathbf{q},t)}(t|\widehat{\rho}_{\mathbf{q}}^{(0)}|t). \tag{14}$$

Then $\mathbf{r}_0(\mathbf{q},t)$ must be approximately equal to the mean position of the particle at time t. We define the deviation of the current from the "zeroth" value to be

$$\Delta\hat{\mathbf{j}}_{\mathbf{q}}(t) = \hat{\mathbf{j}}_{\mathbf{q}}(t) - \hat{\mathbf{j}}^{(0)}_{\mathbf{q}}(t).$$

This deviation will be used to build the interaction operator in the new representation. The above transformation is convenient because the operators (13) commute at different times:

$$[\hat{j}^{(0)a}_{\mathbf{q}}(t), \hat{j}^{(0)b}_{\mathbf{q}'}(t')] = 0. \tag{15}$$

Using (13), we write the electromagnetic interaction operator as a sum of two terms, $\widehat{H}_{int}(t) = \widehat{H}^{(0)}_{int}(t) + \widehat{H}^{(1)}_{int}(t)$, where

$$\widehat{H}^{(0)}_{int}(t) = -\frac{1}{c}\int \hat{\mathbf{j}}^{(0)}(t)\widehat{\mathbf{A}}(t)dV; \tag{16}$$

$$\widehat{H}^{(1)}_{int}(t) = -\frac{1}{c}\int \Delta\hat{\mathbf{j}}(t)\widehat{\mathbf{A}}(t)dV. \tag{17}$$

Then, by virtue of (15), the equation

$$i\frac{d}{dt}|t) = \widehat{H}^{(0)}_{int}(t)|t) \tag{18}$$

has an exact solution in the form of a direct product of the vectors of extended (or modified, in terminology of Ref. 10) coherent states,

$$|t) = \prod_{\alpha,\mathbf{q}} \exp\left(-i\hat{\chi}_{\alpha\mathbf{q}} - \hat{f}_{\alpha\mathbf{q}}\widehat{Q}^{\dagger}_{\alpha\mathbf{q}} + \hat{f}^{\dagger}_{\alpha\mathbf{q}}\widehat{Q}_{\alpha\mathbf{q}}\right)|0), \tag{19}$$

where (at $t_0 = 0$)

$$\widehat{Q}_{\alpha\mathbf{q}}(t) = i\frac{g_q}{c}\int\limits_0^t dt'\mathbf{e}^*_{\alpha\mathbf{q}}\hat{\mathbf{j}}^{(0)}_{\mathbf{q}}(t')e^{i\omega t'}, \tag{20}$$

$$\hat{\chi}_{\alpha\mathbf{q}}(t) = -\frac{i}{2}\int\limits_0^t \{\dot{\widehat{Q}}^{\dagger}_{\alpha\mathbf{q}}(t')\widehat{Q}_{\alpha\mathbf{q}}(t') - \widehat{Q}^{\dagger}_{\alpha\mathbf{q}}(t')\dot{\widehat{Q}}_{\alpha\mathbf{q}}(t')\}dt'. \tag{21}$$

The initial state vector $|0)$ is the direct product of the vacuum state of the electromagnetic field, $|vac)$, and the vector of the initial state of the moving particle, $|\phi)$, described by the wave function $\phi(\mathbf{r}), i.e.,$ $|0) = |\phi, vac)$.

We have chosen $t_0 = 0$ to be zero rather than $-\infty$ due to the fact that, as further calculation show, the temporal sequence of changes in the state of a moving particle that interacts with the field of the radiation it emits is highly important. In this approach there are sure to be problems assosiatad with the inteaction turning on, which violates the charge conservation, and with the generation of virtual radiation, which is the consequence of such violation. To avoid the need to discard fictitious terms, one can resort to turning the interaction on slowly by replacing the constant Z with a slowly increasing charge $Z(1-e^{-\epsilon t})$, where ϵ is small. The charge buildup time $\tau_{in}=\epsilon^{-1}$ must be long compared to ω^{-1}, but short compared to the observation time (here t must be much longer than ω^{-1}). After we establish a method for evaluating the integrals for some definite value of q, we can extend it to any other value of q.

If we ignore the corrections generated by $\widehat{H}^{(1)}_{int}$, Eq. (19) fully solves the problem of calculating the physical quantities of interest. In particular,instead of (5) we have

$$n_{\alpha\mathbf{q}}(t)=(t|\hat{f}^{\dagger}_{\alpha\mathbf{q}}(t)\hat{f}_{\alpha\mathbf{q}}(t)|t),$$

which at $\mathbf{r}_0(\mathbf{q},t)=\mathbf{r}_0(t)$ leads to a result coinciding with (5). Thus, if we ignore $\widehat{H}^{(1)}_{int}$, the current variant of the quantum mechanical theory differs from the semiclassical one in calculations of the mean number of the emitted photons or the energy of these photons only when $\mathbf{r}_0(\mathbf{q},t)$ differs from $\mathbf{r}_0(t)$.

At the same time, corrections due to $\widehat{H}^{(1)}_{int}$, can be obtained for any convenient choise of the vectors $\mathbf{r}_0(\mathbf{q},t)$. Let us put $\mathbf{r}_0(\mathbf{q},t)=\mathbf{r}_0(t)$. Then $\widehat{Q}_{\alpha\mathbf{q}}(t)=Q_{\alpha\mathbf{q}}(t)\hat{\rho}_{\mathbf{q}}$, where $Q_{\alpha\mathbf{q}}(t)$ is specified by its semiclassical expression

$$Q_{\alpha\mathbf{q}}(t)=i\frac{Z}{c}g_q\int\limits_0^t dt'\mathbf{e}^*_{\alpha\mathbf{q}}\mathbf{v}_0(t')\exp\{i\omega t'-i\mathbf{q}\mathbf{r}_0(t')\}. \tag{22}$$

To construct a new "modified" perturbation theory in $\widehat{H}^{(1)}_{int}$, we introduce the zero-order evolution operator

$$\widehat{U}_0(t)=\exp\{\sum_{\alpha,\mathbf{q}}\widehat{Q}_{\alpha\mathbf{q}}(t)\hat{f}^{\dagger}_{\alpha\mathbf{q}}-\widehat{Q}^{\dagger}_{\alpha\mathbf{q}}(t)\hat{f}_{\alpha\mathbf{q}}-i\hat{\chi}_{\alpha\mathbf{q}}(t)\}.$$

Then (19) can be written as $|t)=\widehat{U}_0(t)|0)$. We also introduce a new representation of operators:

$$\widetilde{A}(t)=\widehat{U}^{\dagger}_0(t)\widehat{A}(t)\widehat{U}_0(t). \tag{23}$$

The state vector $|t>$ in this representation obeys the equation

$$i\frac{d}{dt}|t>=\widetilde{H}^{(1)}_{int}(t)|t>. \tag{24}$$

Allowing for (24), we can reduce the expression for the mean number of photons to

$$n_{\alpha\mathbf{q}}(t) = n^{(0)}_{\alpha\mathbf{q}}(t) + \sum_{n=1}^{\infty}(-i)^n \int\limits_0^t dt_1 \int\limits_0^{t_1} dt_2 ... \int\limits_0^{t_{n-1}} dt_n *$$

$$< 0|[...[[\widetilde{f}^{\dagger}_{\alpha\mathbf{q}}(t)\widetilde{f}_{\alpha\mathbf{q}}(t), \widetilde{H}^{(1)}_{int}(t_1)], \widetilde{H}^{(1)}_{int}(t_2)], ..., \widetilde{H}^{(1)}_{int}(t_n)]|0> . \tag{25}$$

4. CALCULATING CORRECTIONS IN THE MODIFIED THEORY

Writting the series in Eq. (25) explicitly, we find that the expansion contains terms proportional to even powers of Z. We collect the leading terms of this type, which contain Z^2 as a pre-exponential factor. Such terms exist in $n^{(0)}_{\alpha\mathbf{q}}(t)$ and in the first and second of the series (25). Note that here Z is also contained in the exponents entering into $\widehat{U}_0$ and $\widehat{U}^{\dagger}_0$. We calculate the first commutator in (25) via the following auxiliary formulas:

$$\widetilde{f}_{\alpha\mathbf{q}}(t) = \left(\hat{f}_{\alpha\mathbf{q}} + \widehat{Q}_{\alpha\mathbf{q}}(t)\right)e^{-i\omega t};\ \ \widetilde{f}^{\dagger}_{\alpha\mathbf{q}}(t) = \left(\hat{f}^{\dagger}_{\alpha\mathbf{q}} + \widehat{Q}^{\dagger}_{\alpha\mathbf{q}}(t)\right)e^{i\omega t};$$

$$[\hat{f}_{\alpha\mathbf{q}}, \widehat{U}_0(t)] = \widehat{U}_0(t)\widehat{Q}_{\alpha\mathbf{q}}(t);\ \ [\hat{f}^{\dagger}_{\alpha\mathbf{q}}, \widehat{U}^{\dagger}_0(t)] = -\widehat{U}^{\dagger}_0(t)\widehat{Q}^{\dagger}_{\alpha\mathbf{q}}(t);$$

$$[\hat{f}^{\dagger}_{\alpha\mathbf{q}}, \widehat{U}_0(t)] = \widehat{U}_0(t)\widehat{Q}^{\dagger}_{\alpha\mathbf{q}}(t);\ \ [\hat{f}_{\alpha\mathbf{q}}, \widehat{U}^{\dagger}_0(t)] = -\widehat{U}^{\dagger}_0(t)\widehat{Q}_{\alpha\mathbf{q}}(t). \tag{26}$$

We put

$$\widehat{B}_{\alpha\mathbf{q}}(t) = \mathbf{e}_{\alpha\mathbf{q}}\Delta\hat{\mathbf{j}}_{-\mathbf{q}}(t),$$

so that

$$\widehat{H}^{(1)}_{int}(t) = -\frac{Z}{c}\sum_{\alpha,\mathbf{q}} g_q\left(\hat{f}_{\alpha\mathbf{q}}\widehat{B}_{\alpha\mathbf{q}}(t)e^{-i\omega t} + \hat{f}^{\dagger}_{\alpha\mathbf{q}}\widehat{B}^{\dagger}_{\alpha\mathbf{q}}(t)e^{i\omega t}\right). \tag{27}$$

Using (26), we can perform th following transformation:

$$[\widetilde{f}^{\dagger}_{\alpha\mathbf{q}}(t)\widetilde{f}_{\alpha\mathbf{q}}(t), \widetilde{H}^{(1)}_{int}(t)] =$$

$$\widehat{U}^{\dagger}_0(t_1)[\left(\hat{f}^{\dagger}_{\alpha\mathbf{q}} + \widehat{Q}^{\dagger}_{\alpha\mathbf{q}}(t,t_1)\right)\left(\hat{f}_{\alpha\mathbf{q}} + \widehat{Q}_{\alpha\mathbf{q}}(t,t_1)\right), \widehat{H}^{(1)}_{int}(t_1)]\widehat{U}_0(t_1), \tag{28}$$

where $\widehat{Q}_{\alpha\mathbf{q}}(t,t_1) = \widehat{Q}_{\alpha\mathbf{q}}(t) - \widehat{Q}_{\alpha\mathbf{q}}(t_1)$. Since the operators $\widehat{Q}_{\alpha\mathbf{q}}(t)$ already contain Z as a factor, the leading terms emerge as a result of the commutation of the photon operators and $\widehat{H}^{(1)}_{int}$:

$$[\widetilde{f}^{\dagger}_{\alpha\mathbf{q}}(t)\widetilde{f}_{\alpha\mathbf{q}}(t), \widetilde{H}^{(1)}_{int}(t_1)] \cong \frac{Z}{c}g_q\widehat{U}^{\dagger}_0(t_1)\Big(e^{-i\omega t_1}\widehat{B}_{\alpha\mathbf{q}}(t_1)(\hat{f}_{\alpha\mathbf{q}} + \widehat{Q}_{\alpha\mathbf{q}}(t,t_1)) -$$

$$e^{i\omega t_1}(\widehat{f}^\dagger_{\alpha\mathbf{q}}+\widehat{Q}^\dagger_{\alpha\mathbf{q}}(t,t_1))\widehat{B}^\dagger_{\alpha\mathbf{q}}(t_1)\Big)\widehat{U}_0(t_1). \tag{29}$$

If we average (29) over the initial state of the system by employing the equalities

$$\widehat{f}_{\alpha\mathbf{q}}\widehat{U}_0(t_1)|0)=\widehat{Q}_{\alpha\mathbf{q}}(t_1)\widehat{U}_0(t_1)|0);$$
$$(0|\widehat{U}^\dagger_0(t_1)\widehat{f}^\dagger_{\alpha\mathbf{q}}=(0|\widehat{U}^\dagger_0(t_1)\widehat{Q}^\dagger_{\alpha\mathbf{q}}(t_1),$$

we get

$$(0|[\widetilde{f}^\dagger_{\alpha\mathbf{q}}(t)\widetilde{f}_{\alpha\mathbf{q}}(t),\widetilde{H}^{(1)}_{int}(t_1)]|0)=\frac{Z}{c}g_q(0|\widehat{U}^\dagger_0(t_1)\{\widehat{B}_{\alpha\mathbf{q}}(t_1)\widehat{Q}_{\alpha\mathbf{q}}(t)e^{-i\omega t_1}-$$
$$e^{i\omega t_1}\widehat{Q}^\dagger_{\alpha\mathbf{q}}(t)\widehat{B}^\dagger_{\alpha\mathbf{q}}(t_1)\}\widehat{U}_0(t_1)|0). \tag{30}$$

In calculating the next corrections in (25) we emmediately discard terms that contain pre-exponential factors with Z raized to a power greater than two. This means that when we plug such terms into the second and subsequent terms of the sum in (25) into the expression for the first-order commutator, we can immediately discard terms containing the operators $\widehat{Q}_{\alpha\mathbf{q}}$ and $\widehat{Q}^\dagger_{\alpha\mathbf{q}}$. In the resulting expressions, the operators $\widehat{f}_{\alpha\mathbf{q}}$ and $\widehat{f}^\dagger_{\alpha\mathbf{q}}$ can be freely interchanged with the operators $\widehat{U}_0$ and $\widehat{U}^\dagger_0$, since their commutators contain heigher-order corrections in Z, which we have just discarded.

In view of this, all terms in which the annihilation operators $\widehat{f}_{\alpha\mathbf{q}}$ are to the right of other $\widehat{f}$- operators, or in which the creation operators $\widehat{f}^\dagger_{\alpha\mathbf{q}}$ are to the left of other $\widehat{f}$-operators, must be dropped. In the remaining terms the operator products $\widehat{f}_{\alpha\mathbf{q}}\widehat{f}^\dagger_{\alpha'\mathbf{q}'}$ must be replaced by the commutators $\delta_{\alpha\alpha'}\Delta(\mathbf{q}-\mathbf{q}')$. By performing these transformations we reduce the leading terms that appear when we write the double commutator on the right hand side of Eq.(25) explicitly to the form

$$-\frac{Z^2}{c^2}g_q^2\Big(\widehat{U}^\dagger_0(t_1)e^{-i\omega(t_1-t_2)}\widehat{B}_{\alpha\mathbf{q}}(t_1)\widehat{U}_0(t_1)\widehat{U}^\dagger_0(t_2)\widehat{B}^\dagger_{\alpha\mathbf{q}}(t_2)\widehat{U}_0(t_2)+$$
$$\widehat{U}^\dagger_0(t_2)\widehat{B}_{\alpha\mathbf{q}}(t_2)\widehat{U}_0(t_2)\widehat{U}^\dagger_0(t_1)\widehat{B}^\dagger_{\alpha\mathbf{q}}(t_1)\widehat{U}_0(t_1)e^{i\omega(t_1-t_2)}\Big). \tag{31}$$

Collecting all terms of the specified order,we get

$$n_{\alpha\mathbf{q}}(t)=|Q_{\alpha\mathbf{q}}(t)|^2-i\frac{Z}{c}g_q\int\limits_0^t dt_1\Big(0|\widehat{U}^\dagger_0(t_1)\{e^{-i\omega t_1}\widehat{B}_{\alpha\mathbf{q}}(t_1)\widehat{Q}_{\alpha\mathbf{q}}(t)-$$
$$e^{i\omega t_1}\widehat{Q}^\dagger_{\alpha\mathbf{q}}(t)\widehat{B}^\dagger_{\alpha\mathbf{q}}(t_1)\}\widehat{U}_0(t_1)|0\Big)+\frac{Z^2}{c^2}g_q^2\int\limits_0^t dt_1\int\limits_0^{t_1} dt_2\Big(0|\widehat{U}^\dagger_0(t_1)e^{-i\omega(t_1-t_2)}*$$

$$\widehat{B}_{\alpha\mathbf{q}}(t_1)\widehat{U}_0(t_1)\widehat{U}_0^{\dagger}(t_2)B_{\alpha\mathbf{q}}^{\dagger}(t_2)\widehat{U}_0(t_2)+$$

$$\widehat{U}_0^{\dagger}(t_2)\widehat{B}_{\alpha\mathbf{q}}^{\dagger}(t_2)\widehat{U}_0(t_2)\widehat{U}_0^{\dagger}(t_1)e^{i\omega(t_1-t_2)}\widehat{B}_{\alpha\mathbf{q}}(t_1)\widehat{U}_0(t_1)|0\Big). \tag{32}$$

If we now write all terms in (32) that appear because of plugging the explicit expressions for $\widehat{B}_{\alpha\mathbf{q}}(t)$ into (32), collect like terms, and do the necessary canceling, we arrive at the final result:

$$n_{\alpha\mathbf{q}}(t) = \frac{Z^2}{c^2}g_q^2\int\limits_0^t dt_1\int\limits_0^t dt_2(0|\widehat{U}_0^{\dagger}(t_1)*$$

$$\mathbf{e}_{\alpha\mathbf{q}}\hat{\mathbf{j}}_{\mathbf{q}}^{\dagger}(t_1)\widehat{U}_0(t_1)\widehat{U}_0^{\dagger}(t_2)\mathbf{e}_{\alpha\mathbf{q}}^{*}\hat{\mathbf{j}}_{\mathbf{q}}(t_2)\widehat{U}_0(t_2)|0)e^{-i\omega(t_1-t_2)}. \tag{33}$$

Note that at deriving (33), we did not take advantage of the fact that $\mathbf{r}_0(\mathbf{q},t)$ is independent of $\mathbf{q}$, with the result that the formula still holds in the general case, in which $\mathbf{r}_0(t)$ is replaced by $\mathbf{r}_0(\mathbf{q},t)$ in(22).

5. NUMBER OF PHOTONS

We assume that in the expansion of the initial state vector of the particle, $|0)$, the expancion coefficients $C_{\mathbf{k}_i}$ in states with definite momentum $|\mathbf{k}_i)$ have a peak at $\mathbf{k}_0$, and decrease as $\mathbf{k}_i$ $\mathbf{k}_0$ devates from $\mathbf{k}_0$, by the Gauss law

$$C_{\mathbf{k}_i} = (2\pi\delta_{\perp}^2)^{1/2}(2\pi\delta_l^2)^{1/4}\exp[-p_i^2\delta_l^2/4-(\mathbf{k}_{i\perp}-\mathbf{k}_{0\perp})^2\delta_{\perp}^2/4],$$

where $\mathbf{k}=(\mathbf{k}_{\perp},p_i)$, $\mathbf{k}_{0\perp}$ is time- dependent, and δ_l and $\delta_{\perp}$ - are longitudinal and transversal packet widths (relative to the z axis). This representation follows from the study of electron states in a magnetic field in the Appendix. For relativistic electrons, the momentum uncertainty in the initial state is much less than the momentum proper. In real calculations of the numbers of emitted photons via (33), it is preferable to represent the current operators as expansions in states with definite momentum at a given moment in time, with a time dependence characteristic of plane waves. In the present paper, this approximation is justified by the fact that due to the strong effect of the radiation on the particle's state in the comoving reference frame, an effect exeeding the one produced by the external field, we can ignore the quantization of levels in the time dependence of the operators.Indeed, even the classical theory of synchrotron radiation predicts that the mean energy of the photons emitted by a particle is much greater than ω_0. In view of this, the mean difference in particle energies before and after photon emission proves to be much greater than the separation between the levels of transverse motion. Under these conditions, allowance for level quantization in the time dependence of the operator can only lead to small corrections of order $1/\overline{n}$ (where $\overline{n} \sim \gamma^3$ - is the mean ratio of the frequency of the emitted photon to ω_0).

As a result oof the action of electron operators, the vectors $\mathbf{k}$ and $\mathbf{k_1}$ in the current operators in (33) are transformed into the vectors $\mathbf{k}_i - \Delta\mathbf{q}$, where $\Delta\mathbf{q} = \sum_s \mathbf{q}_s$, with $\mathbf{q}_1, \mathbf{q}_2, ...$ the momenta of emitted photons. Replacing the given expression with $\mathbf{k}_i(t) = \mathbf{k}_i - \Delta\mathbf{k}(t)$, where $\Delta\mathbf{k}(t)$ is the mean momentum loss by the particle by the time t, and plugging it into all the cofactors in (33) that are not in the exponential, we get

$$n_{\alpha\mathbf{q}}(t) = \frac{Z^2}{c^2} g_q^2 \int_0^t dt_1 \int_0^t dt_2 e^{-i\omega(t_1-t_2)} \sum_{\mathbf{k}_i,\sigma'} |C_{\mathbf{k}_i}|^2 (\mathbf{e}_{\alpha\mathbf{q}} \mathbf{v}^*_{i\sigma'}(\mathbf{q}, t_1))(\mathbf{e}^*_{\alpha\mathbf{q}} \mathbf{v}_{i\sigma'}(\mathbf{q}, t_2)) *$$

$$\Big(vac; \mathbf{k}_i, \sigma | \widehat{U}_0^\dagger(t_1) \widehat{\rho}^\dagger_{\mathbf{q}}(t_1) \widehat{U}_0(t_1) \widehat{U}_0^\dagger(t_2) \widehat{\rho}_{\mathbf{q}}(t_2) \widehat{U}_0(t_2) | \mathbf{k}_i, \sigma; vac\Big), \qquad (34)$$

where

$$\widehat{\rho}_{\mathbf{q}}(t) = \sum_{\mathbf{k},\sigma,\sigma'} \widehat{d}^\dagger_{\mathbf{k}-\mathbf{q},\sigma'} \widehat{d}_{\mathbf{k},\sigma} \exp\{i(\varepsilon_{\mathbf{k}-\mathbf{q}} - \varepsilon_{\mathbf{k}})t\},$$

$$v^a_{i\sigma'}(\mathbf{q}, t) = \frac{c^2}{2\sqrt{\varepsilon_i \varepsilon_i'}} w^*_{\sigma'} \left[\sqrt{\frac{\varepsilon_i' + mc^2}{\varepsilon_i + mc^2}} \sigma^a \sigma^b k_i^b(t) + \sqrt{\frac{\varepsilon_i + mc^2}{\varepsilon_i' + mc^2}} \sigma^b (k_i^b(t) - q^b)) \sigma^a \right] w_\sigma,$$

with $\varepsilon_i = \varepsilon_{\mathbf{k}_i(t)}$, $\varepsilon_i' = \varepsilon_{\mathbf{k}_i(t)-\mathbf{q}}$, and the summation over repeated indices is implied. The term coresponding to $\sigma' \neq \sigma$ describes emission processes accompanied by electron spin flip. Further simplification is possible if the exponents in the density operators in (34) are transformed according to

$$\varepsilon_{\mathbf{k}-\mathbf{q}} - \varepsilon_{\mathbf{k}} \approx \sum_s \mu(\mathbf{q}, \mathbf{q}_s'), \qquad (35)$$

where $\mu(\mathbf{q}, \mathbf{q}_s')$ are unspecified functions. In this approach, different photons are assumed to be almost independent, since otherwise we would have to speak of a strong correlations between the emission of two separate photons, which agrees neither with the semiclassical theory nor with the calculations below. In an approximation that is linear in $\Delta\mathbf{q}$, for $q \ll k_i$ we have

$$\mu_i(\mathbf{q}, \mathbf{q}_s') \approx (\nabla\varepsilon_{\mathbf{k_i}-\mathbf{q}} - \nabla\varepsilon_{\mathbf{k_i}})\mathbf{q}_s' \approx -\mathbf{q}\mathbf{q}_s'/m\gamma_i, \qquad (36)$$

where $\gamma_i = \varepsilon_{\mathbf{k}_i}/mc^2$. As $q_s' \to \infty$, the function $\mu(\mathbf{q}\mathbf{q}_s')$ ceases to depend on q_s'.

Using the methods of calculating means employed in Ref.9, we get

$$n_{\alpha\mathbf{q}}(t) = \int\limits_0^t dt_1 \int\limits_0^t dt_2 \sum_{\mathbf{k}_i,\sigma'} |C_{\mathbf{k}_i}|^2 \dot{Q}^*_{i\alpha\mathbf{q}}(t_1,\sigma')\dot{Q}_{i\alpha\mathbf{q}}(t_2,\sigma')\exp[-P_{i\mathbf{q}}(t_1,t_2)], \quad (37)$$

where

$$Q_{i\alpha\mathbf{q}}(t,\sigma') = i\frac{Z}{c}g_q \int\limits_0^t \mathbf{e}^*_{\alpha\mathbf{q}}\mathbf{v}_{i,\sigma'}(\mathbf{q},t')\exp[i\omega t' - i\mathbf{q}\mathbf{r}_{i,\sigma'}(\mathbf{q},t')]dt',$$

with $\dot{\mathbf{r}}_{i,\sigma'}(\mathbf{q},t) = \mathbf{v}_{i,\sigma'}(\mathbf{q},t)$. The exponent in (37) is given by

$$\begin{aligned} P_{i\mathbf{q}}(t_1,t_2) = \sum_{\beta,\mathbf{q}',\sigma'} \Big[& |Q_{i\beta\mathbf{q}'}(t_1,\sigma')|^2 \left(1-\exp[-i\mu_i(\mathbf{q},\mathbf{q}')t_1]\right) + \\ & |Q_{i\beta\mathbf{q}'}(t_2,\sigma')|^2 \left(1-\exp[i\mu_i(\mathbf{q},\mathbf{q}')t_2]\right) - \\ & Q^*_{i\beta\mathbf{q}'}(t_1,\sigma')Q_{i\beta\mathbf{q}'}(t_2,\sigma') \left(1-\exp[-i\mu_i(\mathbf{q},\mathbf{q}')t_1]\right) * \\ & \left(1-\exp[i\mu_i(\mathbf{q},\mathbf{q}')t_2]\right) \Big]. \end{aligned} \quad (38)$$

Obviously,

$$P^*_{i\mathbf{q}}(t_1,t_2) = P_{i\mathbf{q}}(t_2,t_1);$$
$$\lim_{q\to 0} P_{i\mathbf{q}}(t_1,t_2) \to 0;$$
$$\lim_{t_1\to t_2} P_{i\mathbf{q}}(t_1,t_2) \to 0.$$

Equation (37) contains the desired corrections to the semiclassical expression for the number of emitted photons. It assumes its semiclassical form for $|P_{i\mathbf{q}}(t_1,t_2)| \ll 1$. From a physical standpoint, this difference betwwen the formulas is due to the fact that in (37) we allow for interaction of the emitted photons, while in the semiclassical theory this factor is ignored. The probability distribution for the number of emitted photons in each state does not obey the Poisson law any longer, which a reflection of the nonlinearity of electromagnetic phenomena in the quantum theory.

Obviously, an equation like (37) can be used to study arbitrary motion of a particle, not just an electron in a synchrotron. To do so, we merely redefine the quantities $\mathbf{v}_{i\sigma'}(\mathbf{q},t)$, which in the simplex case can be approximately calculated for the mean of the vector $\mathbf{k}_i$ and averajed over spin (in this case, the velocities $\mathbf{v}(\mathbf{q},t)$ and the function (38) no longer depend on the indices i and σ').

Let us estimate $P_{\mathbf{q}}$ for the case in which the velocity $\mathbf{v}(\mathbf{q},t)$ is constant and equal to $\mathbf{v}_0$:

$$Q_{\alpha\mathbf{q}}(t) = \frac{Z}{c} g_q \frac{\mathbf{e}^*_{\alpha\mathbf{q}}\mathbf{v}_0}{\omega - \mathbf{q}\mathbf{v}_0} e^{i(\omega-\mathbf{q}\mathbf{v}_0)t}.$$

Plugging this into (38), we obtain an expression that is logaritmically divergent, due to the slow decrease in the integrands as $q' \to \infty$. This fact is the manifestation of ultraviolet divergence, often encountered in electrodynamics. In contrast to Feinman's perturbation theory, ultraviolet divergence does not lead to a catastrophe: it only means that (in contrast to the predictions of the semiclassical theory) a uniformly moving particleis not accompanied by transverse photons. This example is a clear demonstration of the dependence of the way in which the ultraviolet divergence depends on the perturbation theory employed. A detailed study of this problem lies outside the scope of the present paper, where we use the standard method of introducing a cut-off momentum $q_c \sim mc$ to remove the ultraviolet singularity. The resulting expression for the absolute value of the function (38) proves to be small and varies very slowly (logaritmically) with t_1 and t_2. An explicit estimate of the function (38) for $\mathbf{v}(\mathbf{q},t)$ constant will be made in the next section.

6. INFRARED ASYMPTOTIC BEHAVIOUR OF THE NUMBER OF PHOTONS

Let us consider the asymptotic behaviour of the function (37) as $\omega = qc \to 0$. In classical electrodynamics (see, e.g., Ref. 11) and in the semiclassical theory there is a characteristic frequency dependence of $n_{\alpha\mathbf{q}}$ as $\omega \to 0$, namely, $n_{\alpha\mathbf{q}} \sim 1/\omega^3$. Hence, upon integration with respect to momenta, the total number of emitted photons diverges logaritmically at the lower limit. Will allowing for the effect of emission on a state of the emitting particle (as in Eq. (37)) influence this pattern? To ansver this question, we examing a model problem in which a charged particle moves at constant velocity $\mathbf{v}_1$ and, colliding at time $t_3 > 0$ with a point scatterer, suddenly changes its own velocity by a small quantity $\Delta\mathbf{v} = \mathbf{v}_2 - \mathbf{v}_1$, $|\Delta\mathbf{v}| \ll v_1$, and then proceeds to move at constant velocity $\mathbf{v}_2$. The requirement that this jump in velocity be small simplifies all calculations considerably. Moreover, since a jump in velocity implies infinite acceleration, various nonphysical effects are to be expected. The requirement that the velocity jump be small makes the velocity almost a continuous function, so that such effects can be ignored. When Eq. (37) is employed in calculations, there is the problem of the interaction suddenly turning on at the initial moment in time, which violates charge conservation, and of generation of fictitious radiation, which is the consequence of such violation. To avoid the need to discard fictitious terms, one can use the procedure developed in Sec. 3 to turn the interaction on slowly.

Since the particle is assumed to have a definite velocity, we drop the subscript i in (38) and replace the vectors $\mathbf{v}_{i\sigma}$ with the current value of the velocity. We calculate the resulting functions $P_{\mathbf{q}}(t_1, t_2)$, assuming that $\tau_{in} \ll t_3 \ll t$. To this end, we first estimate the quantities $f(t_1, t_2) = \exp[\pm i\mathbf{q}\mathbf{q}' t_{1,2}/m\gamma]$ on the right-hand side of Eq. (38). Since a photon is emitted when the particle changes velocity, we consider the neighbourhood of the point $t_1 = t_3$, $t_2 = t_3$, assuming that $t_3 \sim \gamma_0 m/q^2$. The vector $\mathbf{q}'$ is the momentum transferred from the moving particle to the emitted quanta of electromagnetic field (photons). The mean value of this momentum is of order $m|\Delta\mathbf{v}|$, so that at $q \ll m|\Delta\mathbf{v}|$ the ratio q'/q can be large. Thus, the absolute value of the exponent in $f(t_1, t_2)$ in the range of parameters under investigation is large, and the exponentials are rapidly varying functions that make a negligible contribution to (38). Eliminating these contributions from the outset, we reduce (38) to the simpler form

$$P_{\mathbf{q}}(t_1, t_2) = \sum_{\beta,\mathbf{q}'} [|Q_{\beta\mathbf{q}'}(t_1)|^2 + |Q_{\beta\mathbf{q}'}(t_2)|^2 -$$

$$Q^*_{\beta\mathbf{q}'}(t_1) Q_{\beta\mathbf{q}'}(t_2) \left(1 + e^{-i\mathbf{q}\mathbf{q}'(t_1 - t_2)/m\gamma}\right)]. \quad (39)$$

We now calculate the function (39) explicitly for $\mathbf{v}(\mathbf{q}, t) = \mathbf{v}_0 = const$. In this case, assuming that q_c is much less than the mean momentum of the emitting particle, we calculate the integral with respect to q' and obtain

$$P_{\mathbf{q}}(t_1, t_2) = \frac{Z^2}{4\pi^2 c^3} \int do' \frac{[\mathbf{n}' \times \mathbf{v}_0]^2}{(1 - \mathbf{n}'\mathbf{v}_0/c)^2} \Big(i\,\mathrm{Si}(\omega_2(t_1 - t_2)) +$$

$$i\,\mathrm{Si}((\omega_2 + \omega_1)(t_1 - t_2)) + 2\widetilde{C} - \mathrm{Ci}(\omega_2 |t_1 - t_2|) -$$

$$\mathrm{Ci}(|\omega_2 + \omega_1||t_1 - t_2|) + \ln\left(\omega_2 |\omega_2 + \omega_1| (t_1 - t_2)^2\right)\Big), \quad (40)$$

where $\mathbf{n}' = \mathbf{q}'/q'$, $\omega_1 = q_c \mathbf{n}'\mathbf{q}/m\gamma$, $\omega_2 = (c - \mathbf{n}'\mathbf{v}_0) q_c$, $\mathrm{Si}(\xi)$ and $\mathrm{Ci}(\xi)$ - are the sine and cosine integrals, and $\widetilde{C} = 0.5772\cdots$ - is Euler's constant. The function (40) vanishes at $t_1 = t_2$ and slowly increases with the time difference $\Delta t = |t_1 - t_2|$. In the nonrelativistic limit at large $\Delta t \gg 1/cq_c$, the function

(40) can be arrpoximated by the expression

$$P_{\mathbf{q}}(t_1,t_2) \approx \frac{2Z^2v_0^2}{3\pi c^3}\left[i\pi\,\mathrm{sign}(t_1-t_2) + 2\left(\widetilde{C} + \ln(cq_c) + \ln|t_1-t_2|\right)\right]. \qquad (41)$$

We remark on the smallness of the coefficient of the expression in square brackets. As Δt increases, the real part of (41) increases logarithmically, but the characteristic buildup time proves to be exponentially large, so that the function (41) can be considered small over the entire range of its arguments.

Now let us estimate the number of photons emitted by the electron in the entire course of its motion for the nonrelativistic case. Integrating by parts, we find, for instance, that

$$n_{\alpha\mathbf{q}}(t) = -i\frac{Z^2}{c^2}g_q^2\int\limits_0^t dt_1 \mathbf{e}_{\alpha\mathbf{q}}\mathbf{v}(t_1)e^{-i\omega t_1 + i\mathbf{q}\mathbf{r}(t_1)} *$$

$$\Big[\frac{\mathbf{e}^*_{\alpha\mathbf{q}}\mathbf{v}(t_2)}{\omega - \mathbf{q}\mathbf{v}(t_2) + i\partial P_{\mathbf{q}}(t_1,t_2)/\partial t_2}\exp\{i\omega t_2 - i\mathbf{q}\mathbf{r}(t_2) - P_{\mathbf{q}}(t_1,t_2)\}\Big|_{t_2=0}^{t_2=t} -$$

$$\int\limits_0^t dt_2 \exp\{i\omega t_2 - i\mathbf{q}\mathbf{r}(t_2) - P_{\mathbf{q}}(t_1,t_2)\}\frac{\partial}{\partial t_2}\left(\frac{\mathbf{e}^*_{\alpha\mathbf{q}}\mathbf{v}(t_2)}{\omega - \mathbf{q}\mathbf{v}(t_2) + i\partial P_{\mathbf{q}}(t_1,t_2)/\partial t_2}\right)\Big].$$

Allowance for the value of the first term inside the square brackets at the lower limit is unjustified because of the violation of charge conservation at $t \to 0$. If we turn the interaction on slowly, then this contribution is zero. First we integrate by parts with respect to t_1, using the same ideas that we used in integrating with respect to t_2. We obtain

$$n_{\alpha\mathbf{q}}(t) \approx \frac{Z^2}{c^2}g_q^2\left|\frac{\mathbf{e}_{\alpha\mathbf{q}}\mathbf{v}_2}{\omega - \mathbf{q}\mathbf{v}_2 + i\partial P_{\mathbf{q}}(t,t_2)/\partial t_2}\Big|_{t_2=t}\right|^2 +$$

$$\frac{Z^2}{c^2}g_q^2\int\limits_0^t dt_2\int\limits_0^t dt_1 \exp\{i\omega(t_2-t_1) + i\mathbf{q}(\mathbf{r}(t_1)-\mathbf{r}(t_2)) - P_{\mathbf{q}}(t_1,t_2)\} *$$

$$\frac{\partial}{\partial t_1}\Big[\frac{\mathbf{e}_{\alpha\mathbf{q}}\mathbf{v}(t_1)}{\omega - \mathbf{q}\mathbf{v}(t_1) - i\partial P_{\mathbf{q}}/\partial t_1}\frac{\partial}{\partial t_2}\left(\frac{\mathbf{e}^*_{\alpha\mathbf{q}}\mathbf{v}(t_2)}{\omega - \mathbf{q}\mathbf{v}(t_2) + i\partial P_{\mathbf{q}}/\partial t_2}\right)\Big], \qquad (42)$$

where we have discarded the rapidly oscillating terms, which contribute nothing to the overall expression for the number of emitted photons. The first term on the right-hand side of Eq.(42) corresponds to the part of the transverse field that follows the moving particle, and is related neither to change in the particle's velocity nor to the radiation. Hence in all calculations of the characteristics of the radiation that follow, we allow only for the second (integral) term.

In calculating the time derivatives in (42) we encounter continuous and delta-function terms, with the latter being a reflection of the discontinuity in velocity, the derivatives $\dot{Q}_{\alpha\mathbf{q}}$ and $\dot{P}_{\mathbf{q}}$. For instance,

$$\frac{\partial}{\partial t_2}\left(\frac{\mathbf{e}^*_{\alpha\mathbf{q}}\mathbf{v}(t_2)}{\omega-\mathbf{qv}(t_2)+i\partial P_{\mathbf{q}}(t_1,t_2)/\partial t_2}\right)=$$

$$i\Theta(t_3-t_2)\frac{\partial^2 P_{\mathbf{q}}(t_1,t_2)}{\partial t_2^2}\frac{\mathbf{e}^*_{\alpha\mathbf{q}}\mathbf{v}_1}{(\omega-\mathbf{qv}_1+i\partial P_{\mathbf{q}}(t_1,t_2)/\partial t_2)^2}+$$
$$i\Theta(t_2-t_3)\frac{\partial^2 P_{\mathbf{q}}(t_1,t_2)}{\partial t_2^2}\frac{\mathbf{e}^*_{\alpha\mathbf{q}}\mathbf{v}_2}{(\omega-\mathbf{qv}_2+i\partial P_{\mathbf{q}}(t_1,t_2)/\partial t_2)^2}+$$
$$\delta(t_2-t_3)\Big[\frac{\mathbf{e}^*_{\alpha\mathbf{q}}\mathbf{v}_2}{\omega-\mathbf{qv}_2+i\partial P_{\mathbf{q}}(t_1,t_2)/\partial t_2}\Big|_{t_2=t_3+0}-$$
$$\frac{\mathbf{e}^*_{\alpha\mathbf{q}}\mathbf{v}_1}{\omega-\mathbf{qv}_1+i\partial P_{\mathbf{q}}(t_1,t_2)/\partial t_2}\Big|_{t_2=t_3-0}\Big]. \tag{43}$$

Here $\Theta(\xi)$ is the Heaviside step function.

The relationship between the continuous and delta-function terms in (43) can be evaluated as follows. The total contribution of the Θ-functions can again be calculated by parts, which again results in a delta-function contribution multiplied by the magnitude of the discontinuity of the integrand at $t_2=t_3$. This jump includes the second derivative of $P_{\mathbf{q}}$ as a factor whose order of magnitude can be estimated to be the product of the first derivative and the mean value of the frequency of the emitted photon. The latter cannot exceed the energy lost by the moving particle, and it is therefore proportional to the small parameter $\lambda=\mathbf{v}_1\Delta\mathbf{v}/v_1^2$. Clearly, allowing for the continuous terms in (43) would mean allowing for the next terms in the series expansion of the integrals in λ. the leading term is still the contribution of the delta function, the only contribution we consider.

Using the condition that the interactionis turned on slowly, we find that

$$Q_{\alpha\mathbf{q}}(t)=\frac{Z}{c}g_q\,\frac{\mathbf{e}^*_{\alpha\mathbf{q}}\mathbf{v}_1}{\omega-\mathbf{qv}_1}e^{i(\omega-\mathbf{qv}_1)t},\ \tau_{in}\ll t\le t_3.$$

For $t_3<t$ the result is different:

$$Q_{\alpha\mathbf{q}}(t)=\frac{Z}{c}g_q\Big[\left(\frac{\mathbf{e}^*_{\alpha\mathbf{q}}\mathbf{v}_1}{\omega-\mathbf{qv}_1}-\frac{\mathbf{e}^*_{\alpha\mathbf{q}}\mathbf{v}_2}{\omega-\mathbf{qv}_2}\right)e^{i(\omega-\mathbf{qv}_1)t_3}+\frac{\mathbf{e}^*_{\alpha\mathbf{q}}\mathbf{v}_2}{\omega-\mathbf{qv}_2}e^{i(\omega-\mathbf{qv}_2)t}\Big].$$

For $\tau_{in} \ll t_2 \leq t_3$ we have

$$\frac{\partial P_{\mathbf{q}}(t_1,t_2)}{\partial t_2} = -i\sum_{\beta\mathbf{q}'} Q^*_{\beta\mathbf{q}'}(t_1)\Big[\frac{Z}{c}g_{q'}(\mathbf{e}^*_{\beta\mathbf{q}'}\mathbf{v}_1)e^{i(\omega-\mathbf{q}\mathbf{v}_1)t_2}(1+$$

$$\exp\{-i\mathbf{q}\mathbf{q}'(t_1-t_2)/m\gamma\}) + \frac{\mathbf{q}\mathbf{q}'}{m\gamma}Q_{\beta\mathbf{q}'}(t_2)\exp\{-i\mathbf{q}\mathbf{q}'(t_1-t_2)/m\gamma\}\Big].$$

Finally, for $t_2 > t_3$ we have

$$\frac{\partial P_{\mathbf{q}}(t_1,t_2)}{\partial t_2} = \sum_{\beta\mathbf{q}'}\Big[-2\frac{Z^2}{c^2}g^2_{q'}\sin\{(\omega'-\mathbf{q}'\mathbf{v}_2)\frac{t_2-t_3}{2}\}(\mathbf{e}^*_{\beta\mathbf{q}'}\mathbf{v}_2)\Big(\frac{\mathbf{e}_{\beta\mathbf{q}'}\mathbf{v}_1}{\omega'-\mathbf{q}'\mathbf{v}_1}-$$

$$\frac{\mathbf{e}_{\beta\mathbf{q}'}\mathbf{v}_2}{\omega'-\mathbf{q}'\mathbf{v}_2}\Big) - i\frac{\mathbf{q}\mathbf{q}'}{m\gamma}Q^*_{\beta\mathbf{q}'}(t_1)Q_{\beta\mathbf{q}'}(t_2)\exp\{-i\mathbf{q}\mathbf{q}'(t_1-t_2)/m\gamma\}-$$

$$iQ^*_{\beta\mathbf{q}'}(t_1)\frac{Z}{c}g_{q'}(\mathbf{e}^*_{\beta\mathbf{q}'}\mathbf{v}_2)e^{i(\omega'-\mathbf{q}'\mathbf{v}_2)t_2}(1+\exp\{-i\mathbf{q}\mathbf{q}'(t_1-t_2)/m\gamma\})\Big].$$

Note that $\partial P_{\mathbf{q}}(t_1,t_2)/\partial t_2$ is continuous in t_1.

Let us calculate the delta-function contribution to the integrals with respect to t_2 in (42), letting $t \to \infty$:

$$n_{\alpha\mathbf{q}}(\infty) = \frac{Z^2}{c^2}g^2_q\int_0^\infty dt_1\exp[-i\omega(t_1-t_3)+i\mathbf{q}(\mathbf{r}_0(t_1)-\mathbf{r}_0(t_3))-P_{\mathbf{q}}(t_1,t_3)]*$$

$$\frac{\partial}{\partial t_1}\Big[\frac{\mathbf{e}_{\alpha\mathbf{q}}\mathbf{v}(t_1)}{\omega-\mathbf{q}\mathbf{v}(t_1)-i\partial P_{\mathbf{q}}(t_1,t_3)/\partial t_1}\Big(\frac{\mathbf{e}^*_{\alpha\mathbf{q}}\mathbf{v}_2}{\omega-\mathbf{q}\mathbf{v}_2+i\partial P_{\mathbf{q}}(t_1,t_2)/\partial t_2}\Big|_{t_2=t_3+0}-$$

$$\frac{\mathbf{e}^*_{\alpha\mathbf{q}}\mathbf{v}_1}{\omega-\mathbf{q}\mathbf{v}_1+i\partial P_{\mathbf{q}}(t_1,t_2)/\partial t_2}\Big|_{t_2=t_3-0}\Big)\Big].$$

Now we integrate with respect to t_1, again limiting ourselves to delta-function contributions. Allowing for the fact that $P_{\mathbf{q}}(t_2,t_1) = P^*_{\mathbf{q}}(t_1,t_2)$, we obtain

$$n_{\alpha\mathbf{q}}(\infty) = \frac{Z^2}{c^2}g^2_q\Big|\frac{\mathbf{e}^*_{\alpha\mathbf{q}}\mathbf{v}_2}{\omega-\mathbf{q}\mathbf{v}_2+i\partial P_{\mathbf{q}}(t_3,t_2)/\partial t_2}\Big|_{t_2=t_3+0}-$$

$$\frac{\mathbf{e}^*_{\alpha\mathbf{q}}\mathbf{v}_1}{\omega-\mathbf{q}\mathbf{v}_1+i\partial P_{\mathbf{q}}(t_3,t_2)/\partial t_2}\Big|_{t_2=t_3-0}\Big|^2. \tag{44}$$

This equation solves the problem. If we neglect the derivatives of $P_{\mathbf{q}}$ in the denominators, (44) coincides with the standard expression for the number of low-frequency photons emitted in a collision, the expression that can be derived in classical electrodynamics[11] and in quantum electrodynamics if we use standard perturbation theory[6]. Let us estimate the derivatives of $P_{\mathbf{q}}$ in the denominators in (44). We have

$$\frac{\partial P_{\mathbf{q}}(t_3,t_2)}{\partial t_2}\Big|_{t_2=t_3+0} = -2i\frac{Z^2}{c^2}\sum_{\mathbf{q}'} g_{q'}^2 \frac{[\mathbf{q}'\times\mathbf{v}_1][\mathbf{q}'\times\mathbf{v}_2]}{q'^2(\omega'-\mathbf{q}'\mathbf{v}_1)} + O_1(q),$$

where $O_1(q)$ is of the first order in q . For q small, noting that

$$\frac{\partial P_{\mathbf{q}}(t_3,t_2)}{\partial t_2}\Big|_{t_2=t_3-0} = -2i\frac{Z^2}{c^2}\sum_{\mathbf{q}'} g_{q'}^2 \frac{[\mathbf{q}'\times\mathbf{v}_1]^2}{q'^2(\omega'-\mathbf{q}'\mathbf{v}_1)} + O_1(q),$$

and that $\mathbf{v}_2 \approx \mathbf{v}_1$, we obtain

$$n_{\alpha\mathbf{q}}(\infty) = \frac{Z^2}{c^2} g_q^2 \left| \frac{\mathbf{e}_{\alpha\mathbf{q}}\mathbf{v}_2}{\omega-\mathbf{q}\mathbf{v}_2+\Delta} - \frac{\mathbf{e}_{\alpha\mathbf{q}}\mathbf{v}_1}{\omega-\mathbf{q}\mathbf{v}_1+\Delta}\right|, \tag{45}$$

where

$$\Delta = 2\frac{Z^2}{c^2}\sum_{\mathbf{q}'} g_{q'}^2 \frac{[\mathbf{q}'\times\mathbf{v}_1]^2}{q'^2(\omega'-\mathbf{q}'\mathbf{v}_1)}. \tag{46}$$

In the nonrelativistic limit $v_0 \ll c$, from (46) we obtain $\Delta \approx 4Z^2 v_1^2 q_c/3\pi c^2$. Equation (45) does not contain the infrared singularity. A reviation from the $n_{\alpha\mathbf{q}} \sim 1/\omega^3$ law with decreasing ω begins at an energy of order Δ. The lower the energy of relative motion of the charged particle and point scatterer, the lowe the aforementioned energy. This estimate also holds if the velocity of the particle changed not suddenly but over a time interval that is short compared to the time of production of a low energy photon.

7. CALCULATING THE DENSITY MATRIX FOR THE CASE OF SYNCHROTRON RADIATION

We now use the above approach to calculate the density martrix of an emitting particle. The exact expression for the density matrix in the representation realized by the transformation (23) has the form

$$\gamma(\mathbf{x},\mathbf{x}',t) =< t|\widetilde{\psi}^{\dagger}(\mathbf{x},t)\widetilde{\psi}(\mathbf{x}',t)|t> . \tag{47}$$

We calculate (47) in the first approximation, replacing the vector $|t>$ by the initial state vector $|0>$. Using the Baker-Hausdorff rule with proper transformation of the evolution operators $\widehat{U}_0(t)$ and $\widehat{U}_0^{\dagger}(t)$, and the operators $\widehat{Q}_{\alpha\mathbf{q}}(t)$ in the form (20), we easily find that

$$\gamma(\mathbf{x},\mathbf{x}',t) = \gamma_0(\mathbf{x},\mathbf{x}',t)\,\exp[-S(\mathbf{x}-\mathbf{x}',t)], \tag{48}$$

where $\gamma_{\mathbf{0}}(\mathbf{x},\mathbf{x}',\mathbf{t}) = \phi^*(\mathbf{x},\mathbf{t})\phi(\mathbf{x}',\mathbf{t})$ is the value of the density matrix tthat does not account for emission and is determined by the wave function $\phi(\mathbf{x},t)$ of the exactly described state of the electron in an external magnetic field. The function $S(\mathbf{x}-\mathbf{x}',t)$ in the exponent is given by

$$S(\mathbf{x}-\mathbf{x}',t) = \sum_{\mathbf{q},\alpha} |Q_{\alpha\mathbf{q}}(t)|^2[1-e^{i\mathbf{q}(\mathbf{x}-\mathbf{x}')}] \tag{49}$$

which vanishes at $\mathbf{x}=\mathbf{x}'$. As $|\mathbf{x}-\mathbf{x}'| \to \infty$, the function (49) acquires the maximum value, equal to the total number of photons emitted by the given moment in time.

The mean momentum of the particle can be evaluated as follows:

$$\overline{\mathbf{p}}(t) = \mathbf{p_0} + i\int |\phi(\mathbf{x},t)|^2\nabla' S(\mathbf{x},\mathbf{x}',t)|_{\mathbf{x}=\mathbf{x}'}\, d^3x. \tag{50}$$

This means that the gradient $\nabla' S(\mathbf{x},\mathbf{x}',t)$ determines the rate of decrease of the mean particle momentum due to emission of photons. If the initial state was stationary, $|\phi(\mathbf{x},t)|^2$ does not depend on time. In this case, the mean force acting on the particle is

$$\mathbf{F}_b = i\int |\psi_0(\mathbf{x},t)|^2\nabla'\dot{S}(\mathbf{x},\mathbf{x}',t)|_{\mathbf{x}'=\mathbf{x}}\, d^3x. \tag{51}$$

The calculation of the function $S(\mathbf{x},\mathbf{x}',t)$ for the case of synchrotron radiation is similar to the calculation of the proton production rate in Sec.4. Noting that S actually depends on the difference $\mathbf{r}=\mathbf{x}-\mathbf{x}'$, we obtain the value of s averaged over one period:

$$\overline{S}(\mathbf{r},t) = t\frac{Z^2}{c}\int_0^{\pi} d\theta\sin\theta\sum_{n=1}^{\infty} n\omega_0\Big[\cot^2\theta\; J_n^2(\frac{nv_0}{c}\sin\theta)+$$

$$\frac{v_0^2}{c^2}J_n'^2(\frac{nv_0}{c}\sin\theta)\Big]\left(1-J_0(r\frac{n\omega_0}{c}\sin\theta_0\;\sin\theta)\;\;\exp(ir\frac{n\omega_0}{c}\cos\theta_0\;\cos\theta)\right), \tag{52}$$

where θ_0 is the polar angle of the vector $\mathbf{r}$ with respect to the axis perpendicular to the orbital plane, $r=|\mathbf{r}|$. In the ultrarelativistic case the following approximate formula is more convenient:

$$\overline{S}(\mathbf{r},t) = t\frac{2^{2/3}Z^2\omega_0}{c}\int_{\varsigma_0}^{\infty} d\varsigma\varsigma^{1/3}\int_{\theta_-}^{\theta_+} d\theta\sin\theta *$$

$$\{\mathrm{ctg}^2\theta\;\mathrm{Ai}^2[(\varsigma/2)^{2/3}(1-\frac{v_0^2}{c^2}\sin^2\theta)]+$$

$$\frac{v_0^4}{c^4}\frac{2^{2/3}\sin^2\theta}{\varsigma^{2/3}}\,\mathrm{Ai'}^2[(\varsigma/2)^{2/3}(1-\frac{v_0^2}{c^2}\sin^2\theta)]\}*$$

$$\left(1-J_0(\sin\theta_0\,\sin\theta\,\frac{r\varsigma\omega_0}{c})\,\exp(i\cos\theta_0\,\cos\theta\,\frac{r\varsigma\omega_0}{c})\right), \tag{53}$$

where $\varsigma_0=\epsilon^{-3}$, $\theta_-=\pi/2-\epsilon$, $\theta_+=\pi/2+\epsilon$ and $1/\gamma\ll\epsilon\ll 1$; $\mathrm{Ai}(z)$ is the Airy function defined in the Ref.12 and $\mathrm{Ai}'(z)$ is its derivative. Obviously, the imaginary part of the averaged expression for S given by (52) and (53) is zero.

According to Ref.10, the density matrix (48) describes an ensemble of pure states (in the sense of von Newmann), whose properties are determined by the behaviour of $e^{-S(\mathbf{r},t)}$. The expansion of the matrix (48) in the density matrices of these pure states can be written

$$\gamma(\mathbf{x},\mathbf{x}',t)=\int d^3a\Phi^*_{\mathbf{a}}(\mathbf{x},t)\,\Phi_{\mathbf{a}}(\mathbf{x}',t)\,N_{\mathbf{a}}(t), \tag{54}$$

where $\Phi_{\mathbf{a}}$ are the wave functions of the pure states, and $N_{\mathbf{a}}(t)d^3a$ are the probabilities that these states are realized at the given momentum in time. The functions $\Phi_{\mathbf{a}}(\mathbf{x},t)$ are proportional to the products of the wave function $\phi\mathbf{x},t)$ and the wave functions $\chi(\mathbf{x}-\mathbf{a},t)$, where $\chi(\mathbf{x},t)$ is the solution of the integral equation

$$G(\mathbf{r},t)=e^{-S(\mathbf{x}-\mathbf{x}',t)}=\int\chi^*(\mathbf{x}-\mathbf{a},t)\,\chi(\mathbf{x}'-\mathbf{a},t)\,d^3a. \tag{55}$$

But what about the existence and uniqueness of the solution of this equation? If we write (55) in the Fourier representation

$$G_{-\mathbf{q}}(t)=\chi^*_{\mathbf{q}}(t)\chi_{\mathbf{q}}(t),$$

the absolute value of the desired function is uniquely defined, but not the phase. However, this uncertainty is a direct consequence of the translation invariance of Eq.(55), whose general solution, therefore, has the form

$$\chi_{\mathbf{q}}=e^{i\alpha}\sqrt{G_{-\mathbf{q}}} \tag{56}$$

with arbitrary real α. A solution exists if the Fourier transform $G_{\mathbf{q}}$ is a real nonnegative quantity. That it is real follows directly from the fact that $\mathrm{Im}S(\mathbf{r},t)$ is odd and $\mathrm{Re}S(\mathbf{r},t)$ even under insversion; the nonnegativity follows from the fact that $\mathrm{Re}S(\mathbf{r},t)$ increases monotonically with distance r.

The effective size of the localization region for the initial state in the orbital plane is $\delta\rho\sim\sqrt{R/\gamma v_0}$ (see Appendix). The quantity $\delta\rho$ is usually much larger than atomic dimensions. The localization region for the initial state along the magnetic field is infinitely large, which is due to the initial uncertainty in the z-component of the momentum. The latter is obviously determined by the macroscopic parameters of the actual experimental layout.

Radiation can substantially alter the picture, and lead to spatial localization of the emitting particle in a region whose size is of the order of atomic dimensions. To estimate the rate of variation of the widths of the states $\Phi_{\mathbf{a}}(t)$ with the passage of time, the function $S(\mathbf{r}, t)$ was calculated numerically for a set of parameters characteristic of the FIAN–60 synchrotron ($E = 0.68GeV$ and $R = 2m$).

8. MAIN CONCLUSIONS

The perturbation theory developed in this paper has made it possible to establish that certain fundamental problems of quantum electrogynamics are not invariant when the type of perturbation theory is altered. With respect to infrared divergence, this is shown by emloying the simple example of an emitting particle that undergoes a sudden change in velocity. The results have been obtained for the nonrelativistic case, since the study of emitting relativistic particles requires a detailed analysis of the ultraviolet asymptotic behavior of the integrand in (38).

The method of removing ultrfviolet divergences by introducing a cutoff momentum, which was adopted in the present paper, is not covariant under Lorenz transformation, and therefore cannot be used in a consistent relativistic theory. But even preliminary studies show that in the new approach the problem of ultraviolet divergence is not catastrophic, in contrast to the case in ordinary perturbation theory of quantum electrodynamics. It is to be hoped that further research will lead to progress in understanding this problem.

Density matrix calculation have shown that reduction of the spatial dimensions of the localization region for the emitting electrons to atomic dimensions can be achieved over a macroscopically long time interval τ_c of some tenths of a second. Can the present theory be applied to such long times? The situation is complicated by the fact that in the course of one orbital revolution, the particle is subject to a solenoidal electric field that balanced the loss of energy to photon emission. If we assume that this field acts during a time interval so short that it only accelerates the particle's wave packet as a whole and is unable to change the particle's internal parameters substantially, then there is no reason why to do estimates we cannot extend the theory to the entire duration of the particle's motion in the synchrotron.

The time τ_c is much shorter than it takes the packet to spread due to the nonequidistant nature of the spectrum of the transverse-motion levels. What is observed is an anisotropy in the packet's width: the packet is most strongly squeezed perpendicular to the magnetic field, and least strongly parallel to the field. The considerable elongation of the packet in the direction of the magnetic field is obvious.

The posibility of strong spatial localization of the emitting particles means that if the acceleration cycle in the synchrotron is long enough, the motion of the particle can be described to high accuracy by the equations of classical mechanics. Nevertheless, this does not mean that the intensity of the radiation must agree with the prediction of classical electrodynamics. Indeed, a localized state in quantum mechanics is completely different in its properties from a localized state in Newton's classical theory. The justification for using Newton's equations of motion to calculate the paths followed by wave packets is provided by Ehrenfest's theorem, but the decisive factor in calculating the intensity of the radiation is the momentum of the particle, rather than the position. In quantum mechanics, a state with a definite momentum is completely delocalized, and in this way differs substantially from states of type $\Phi_{\mathbf{a}}$. There is thus no way in which we can intuitively interpret calculations of the characteristics of radiation using classical ideas. The characteristic common feature of the formulas derived in this paper is the fact that allowing for the mutual interaction of the emitted photons reduces the radiative intensity. A similar result was obtained by Landau and Pomeranchuk[13], who studied the radiation emitted by charged particles moving in continuous media (the Landau—Pomeranchuk effect). The physics of this phenomenon amount to the fact that random collisions of an emitting particle with particles of the medium can reduce the path length over which the radiative indensity builds up coherently. Something similar is observed when photons are emitted into vacuum: multiple emission of photons can mimic the multiple collisions in a continuous medium that lead to a reduction in radiative intensity.

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APPENDIX: COHERENT STATES OF A RELATIVISTIC ELECTRON IN A UNIFORM MAGNETIC FIELD

Let

$$\mathbf{A} = (-\frac{1}{2}y\mathrm{H}_0, \frac{1}{2}x\mathrm{H}_0, 0),$$

where H_0 is the strength of the magnetic field directed along the z axis. The motion of an electron in such a field, which obeys the Dirac equation, has been the topic of numerous studies (see, e.g., Refs. 14-16). The solution given below differs from the well-known one only in some details.

We introduce the lowering operators for the two independent oscillators:

$$\widehat{a}_1 = \frac{1}{2}\sqrt{m\omega_L}(x + i\widehat{p}_x/m\omega_L - iy + \widehat{p}_y/m\omega_L);$$

$$\widehat{a}_2 = \frac{1}{2}\sqrt{m\omega_L}(x + i\widehat{p}_x/m\omega_L + iy - \widehat{p}_y/m\omega_L),$$

where $\omega_L = |e|\mathrm{H}_0/2mc$ is the Larmor frequency. The frequency of the first independent oscillator is twice the Lormor frequency, $\omega_1 = 2\omega_L$, while the frequency of the second oscillator is zero. The set of lowering and raising operators (which are conjugates of lowering operators) satisfies the standard Bose commutation relations. The operators $\widehat{a}_1$ and $\widehat{a}_1^\dagger$ describe the orbital motion of an electron in a magnetic field, while the operators $\widehat{a}_2$ and $\widehat{a}_2^\dagger$ describe the position, fluctuations and other characteristics of the center of the osculating circular orbit, whose mean radius is R.

We next introduce the matrix operator

$$\widehat{D} = \begin{pmatrix} \widehat{p}_z & -2i\sqrt{m\omega_L}\,\widehat{a}_1 \\ 2i\sqrt{m\omega_L}\,\widehat{a}_1^\dagger & -\widehat{p}_z \end{pmatrix}.$$

The energies of the electron's quantum states are

$$E_\xi = \sqrt{m^2c^4 + p^2c^2 + 4\omega_L mc^2(n_1 + \sigma + \frac{1}{2})}, \qquad (A.1)$$

where the label $\xi = (n_1, n_2, \sigma, p)$ simply indicates the set of quantum numbers in parentheses. The $n_1, n_2| = 0, 1, 2, ...$ label the quantum states of the independent oscillators, with n_1 being the principal quantum number. We denote the projection of momentum on the z axis by p. The discrete variable σ takes two values, $\pm 1/2$, corresponding to two possible projections of spin on the direction of the magnetic field. The bispinor describing a stationary state of an electron in a magnetic field is given by

$$\psi_\xi(\mathbf{r}, t) = \frac{1}{\sqrt{2E_\xi}} \begin{pmatrix} \sqrt{E_\xi + mc^2}\varphi_\xi(\mathbf{r}) \\ \frac{c}{\sqrt{E_\xi + mc^2}}\widehat{D}\varphi_\xi(\mathbf{r}) \end{pmatrix} e^{-iE_\xi t}, \qquad (A.2)$$

where $\varphi_\xi(\mathbf{r})$ is a spinor of the form

$$\varphi_\xi(\mathbf{r}) = e^{ipz}\frac{1}{\sqrt{n_1!n_2!}}(\hat{a}_1^\dagger)^{n_1}(\hat{a}_2^\dagger)^{n_2}\varphi_{0\sigma}(\rho). \qquad (A.3)$$

Here

$$\varphi_{0\sigma}(\rho) = \sqrt{m\omega_L/\pi}\, \exp(-m\omega_L\rho^2/2)\,\chi_\sigma,$$

with

$$\rho^2 = x^2 + y^2,\ \chi^*_{1/2} = (1,0),\ \chi^*_{-1/2} = (0,1),$$

and the normalization length along the z axis is taken equal to unity.

An arbitrary solution of the Dirac equation is a linear combination of bispinors of type (A2). Just what linear combination corresponds to the initial state of an electron in the synchrotron? For standard values of synchrotron parameters (for example, for the FIAN–60 synchrotron), the mean value of n_1 is very large (of order 10^{13}), and if the assumption that photons are emitted largely independently of one another is true, so is Poisson's law. In this case, the expected value of the relative fluctuation of the number n_1 is extremely small, $\lambda = \overline{\Delta n_1}/\overline{n}_1 \sim 10^{-6}$. Hence, essentially all terms of the desired linear combination can be expanded in powers of λ, with the result that

$$E_\xi \approx E_{\sigma p} + (2\omega_L/\gamma_{\sigma p})\Delta n_1,$$

where $E_{\sigma p}$ is the value of E_ξ at $\xi = (\overline{n}_1, \overline{n}_2, \sigma, p)$, and $\gamma_{\sigma p} = E_{\sigma p}/mc^2$. We see that the spectrum is essentially uniformly spaced, with the leveles being separated by the mean orbital frequency $\omega_{\sigma p} = 2\omega_L/\gamma_{\sigma p}$ of the electron about the magnetic fiald. When the relative fluctuation of n_1 is small, we can put $E_\xi \approx E_{\sigma p}$ in all nonexponential factors.

The linear combination corresponding to the above requirements has the form

$$\psi_{\sigma p}(\mathbf{r},t) = \frac{1}{\sqrt{2E_{\sigma p}}}\begin{pmatrix} \sqrt{E_{\sigma p}+mc^2}\varphi_{\sigma p}(\mathbf{r},t) \\ \frac{c}{\sqrt{E_{\sigma p}+mc^2}}\widehat{D}\varphi_{\sigma p}(\mathbf{r},t)\end{pmatrix}\exp(-i\Delta E_{\sigma p}t), \qquad (A.4)$$

where $\Delta E_{\sigma p} = E_{\sigma p} - \omega_{\sigma p}\overline{n}_1$, and

$$\varphi_{\sigma p}(\mathbf{r},t) = e^{ipz}\exp[\sqrt{\overline{n}_1}(e^{i\alpha_1}\hat{a}_1^\dagger(t) - e^{-i\alpha_1}\hat{a}_1(t))+$$

$$\sqrt{\overline{n}_2}(e^{i\alpha_2}\hat{a}_2^\dagger(t) - e^{-i\alpha_2}\hat{a}_2(t))]\varphi_{0\sigma}(\rho),$$

with $\hat{a}_1^\dagger(t) = \hat{a}_1^\dagger \exp(i\omega_{\sigma p}t)$, and $\hat{a}_2^\dagger(t) = \hat{a}_2^\dagger$, where α_1 and α_2 are constant phases; the momentum along the z axis is assumed equal to p. The components of the current density vector in the state (A4) are

$$j_{\sigma p}^{x} = \frac{2c^2}{E_{\sigma p}}\sqrt{m\omega_L \overline{n}_1}|\varphi_{\sigma p}(\mathbf{r},t)|^2 \sin(\omega_{\sigma p}t + \alpha_1)$$

$$j_{\sigma p}^{y} = -\frac{2c^2}{E_{\sigma p}}\sqrt{m\omega_L \overline{n}_1}|\varphi_{\sigma p}(\mathbf{r},t)|^2 \cos(\omega_{\sigma p}t + \alpha_1) \qquad (A.5)$$

$$j_{\sigma p}^{(z)} = -\frac{c^2 p}{E_{\sigma p}}|\varphi_{\sigma p}(\mathbf{r},t)|^2.$$

The packet's rms width in the radial direction in the state (A4) is determined by the radial behavior of the function $\varphi_{0\sigma}(\rho)$ and can be estimated to be $\Delta\rho = \sqrt{2Rc/E_{\sigma p}}$. In the azimutal direction, the stationary states of type (A2) are completely delocalized. Indeed, in these states the angular momentum is well-defined, and by virtue of the uncertainty relation for action-angle variables, they cannot be localized in angle.

In contrast, the state (A4) has no definite angular momentum, but its angular width is limited, and is of order $\Delta\phi \sim 1/\sqrt{\overline{n}_1}$ in the azimutal direction (we assume that the uncertainty in the position of the orbit's center is much smaller than the orbit's radius, so that $n_2 \ll n_1$), which after being multiplied by the orbit's radius yields a distance roughly equal to $\Delta\rho$ (for the FIAN-60 synchrotron this distance is about one micrometer).

The packet width along the z axis is governed by such macroscopic parameters of the device as the diaphragm width, and for this reason it can exceed the radial or azimutal width many times over. In this case the packet can be represented by a linear combination of states of type (A4):

$$\psi_\sigma(\mathbf{r},t) = \sum_p C_p \psi_{\sigma p}(\mathbf{r},t), \qquad (A.6)$$

where the constants C_p satisfy the normalization condition and guarantee, e.g., a Gausian dependence on the z projection of the momentum with midpoint at $p = 0$:

$$C_p = (2\pi\delta_0^2)^{1/4} e^{-p^2\delta_0^2/4}.$$

If we assume that the spatial width of the packet along the z axes is much greater thanthe radial width, then $\delta_0 \gg \Delta\rho$, and in this case the state (A6) is associated with a small symmetric ellipsoid elongated in the direction of the magnetic field and revolving in this orientation ina circular orbit about an axis parallel to z. To estimate the time of packet spread in the radial or azimutal direction, we must keep the next term in the expansion of the energy E_ξ in powers of Δn_1. This yields the value of the time of packet spreading due to the nonequidistant levels of transverse motion, $\tau_1 \sim \gamma_0 R^2/\overline{\Delta n_1}$. Here γ_0 is the Lorenz factor for the electron beam in a synchrotron. For the FIAN–60 synchrotron the time τ_1 was estimated to be about ten seconds.

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